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Research Report

NEURONAL MODELS IN INFINITE-DIMENSIONAL SPACES AND THEIR FINITE-DIMENSIONAL PROJECTIONS: PART I

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Methods of comparing discrete and continuous cable models of single neurons and dynamical phenomena observed in neurobiology can be described with infinite-coupled systems of semilinear parabolic differential-functional equations of the reaction-diffusion-convection type or infinite systems of ordinary integro-differential equations. It is known that numerous problems in computational neuroscience use finite systems of equations based on the so-called compartmental model. It seems a natural idea to extend the results obtained in the theory of finite systems onto infinite systems. However, this requires stringent assumptions to be adopted to achieve compatibility. In most instances the dynamics of infinite systems behave differently to their finite-dimensional projections. The truncation method applied to infinite systems of equations and presented herein yields a truncated system consisting of the first N equations of the infinite system in N unknown functions. A solution of infinite system is defined as the limit when $N \to \infty$ of the sequence of approximations $\{z_N\}_{N=1,2,\ldots}$, where $z_N = (z_N^1, z_N^2, \ldots, z_N^N)$ are defined as solutions of suitable finite truncated systems with corresponding initial-boundary conditions. Geometrically, it may be described as the projection of an infinite system of differential equations considered in a function abstract space of infinite dimension (such as Banach or Hilbert space) onto its finite-dimensional subspaces.

Keywords: Banach space; infinite-countable system; infinite-uncountable system; truncation method; truncated system; projection operator; discrete model; continuous model; cable model; compartmental model; neurons.

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1. Introduction

Nonlinear cable properties of single neurons are ubiquitous to all biological neuronal networks, and their dynamics is understood through the mathematical modeling of nonlinear parabolic differential equations of the reaction-diffusion type. Infinite systems of nonlinear parabolic differential equations of the reaction-diffusion type have been appearing in mathematics in response to problems emerging from neuroscience [3, 11, 13, 40]. Most mathematical models arising from neuroscience involve coupled system of parabolic differential equations referred to as nonlinear reaction-diffusion equations. In computational neuroscience it is accepted that the dependent variable for example, membrane potential does not vary from point to point in space (i.e., remains isopotential), and the transport processes can be ignored. This is known as the compartmental model (see [26, 27, 43]). In the compartmental model, the partial differential equations are replaced with ordinary differential equations. Each compartment can represent an arbitrarily small section of cable or the entire neuron, referred to as a spiking neuron [14]. Spiking neurons have been popularized in mathematical neuroscience through the use of mostly geometrical methods [10, 17]. Regardless of their popularity it has not yet been feasible to use the classical Hopf-bifurcation analysis to determine if spiking neurons undergo through the same bifurcation points as the discrete cables or if co-dimension-2 bifurcations are ubiquitous in neuronal cables. Moreover much of the existing literature has been based on an unquestioning acceptance of the validity of the compartmental model. In fact, Conway *et al.* [9] have shown that the compartmental model is valid approximation only for asymptotic behavior of the solutions of systems of nonlinear reaction-diffusion equations. The subject of the present paper is to use functional analysis methods in order to gauge if the solutions of the equations representing nerve propagation and dendritic spike attenuation differ from those obtained by the compartmental models.

The application of infinite systems of differential equations to describe the dynamics of neurons with the use of cable models assumes that the number of variables involved in the modeling processes is unbounded. This assumption, in turn, leads to cable models involving infinite systems of differential equations. While constructing cable models, there appear two main descriptions of the processes considered: a discrete cable version and a continuous cable version. If a variable taking countable infinite number of values (i.e., circuit of a patch of membrane) is used to describe the process then a discrete cable model of this process is obtained. Discrete cable models are expressed in terms of infinite countable systems of equations. On the other hand, if a continuous space and time variables are used then a continuous cable model is obtained, which are expressed in terms of infinite uncountable systems of equations (see Fig. 1).

Idan Segev wrote [43, p. 94]: "When the membrane properties are voltage-dependent, as is the case with membranes that show rectification or that support action potentials, the analytical approach using linear cable theory is no longer

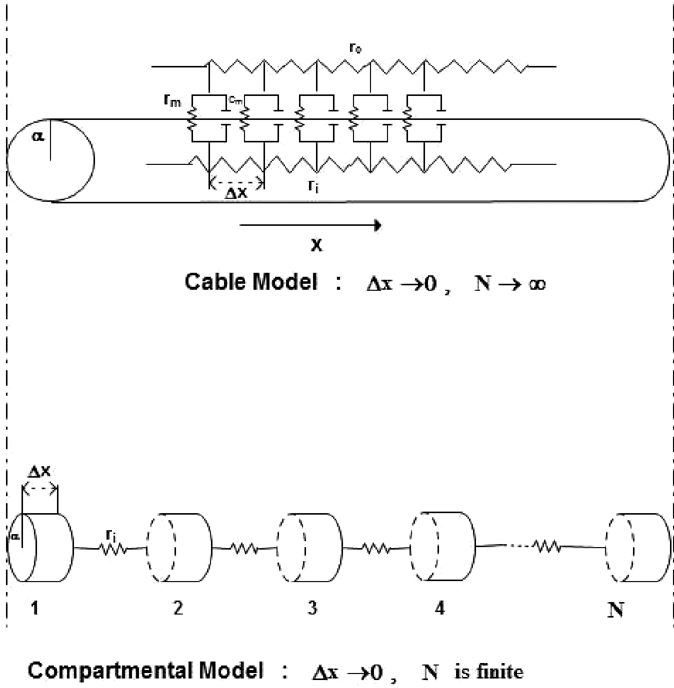

Fig. 1. Discrete and continuous cable models in infinite-dimensional Banach spaces (top) and their finite-dimensional projections (bottom).

valid. As Rall pointed out early on, these complex cases must be dealt with using compartmental rather than analytical models (Rall, 1964)."

In this paper we lay the groundwork to prove otherwise the statement by Segev. The subject of the present paper focuses on problems associated with using the truncation method to investigate neuronal cable models (both continuous and discrete). In the second part of the series, we will utilize comparison theorems and maximum principles for finite and infinite parabolic systems and apply the results to show how neuronal cable models exhibit different dynamics. Both linear and nonlinear cables will be analyzed to indicate that the theory applies to both passive and active neuronal membranes of dendrites. The mathematical analyses lays the foundation in proving that compartmental models are poor exemplars of neuronal dynamics.

2. Notations and Definitions

Let D be a domain in the time-space $(t, x) = (t, x_1, x_2, \ldots, x_m)$ and S be an arbitrary set of indices (finite or infinite).

Let $\mathcal{B}(S)$ be the real space of functions

$$w: S \to \mathbb{R}, \quad j \mapsto w(j) := w^j,$$

such that

$$\sup\{|w^j| : j \in S\} < \infty$$

equipped with the supremum norm

$$\|w\|_{\mathcal{B}(S)} := \sup\{|w^j| : j \in S\}.$$

This space is a Banach space.

We use the symbol $|\cdot|$ to denote the absolute value of a real number, and we write $w = \{w^j\}_{j\in S}$.

The space ℓ^∞ is the sequence space of all real-valued bounded sequences $w = \{w^j\}_{j\in\mathbb{N}} = (w^1, w^2, \ldots)$ such that

$$\sup\left\{|w^j| : j \in \mathbb{N}\right\} < \infty$$

equipped with the norm

$$\|w\|_{\ell^\infty} := \sup\left\{|w^j| : j \in \mathbb{N}\right\}$$

This space is the Banach sequence space.

The partial order "$\leq$" in the space ℓ^∞ is defined by the positive cone

$$\ell^\infty_+ := \{w \colon w = \{w^j\}_{j\in\mathbb{N}} \in \ell^\infty,\ w^j \geq 0 \quad \text{for } j \in \mathbb{N}\}$$

in the following way

$$u \leq v \Leftrightarrow v - u \in \ell^\infty_+.$$

If S is a finite set of indices with r elements, i.e., $S = \{1, 2, \ldots, r\}$ then $\mathcal{B}(S) = \mathbb{R}^r$ and for an infinite countable set S, there is $\mathcal{B}(S) = \mathcal{B}(\mathbb{N}) = \ell^\infty$. For an infinite uncountable set S, there is $\mathcal{B}(S) = \mathcal{B}(\mathbb{R}^\infty)$.

We introduce three spaces of sequences of real-valued functions (see [18]) equipped with the norms induced by the norm of the space ℓ^∞.

Denote by $\mathscr{C}_\mathbb{N}(\bar{D}) := \mathscr{C}^0_\mathbb{N}(\bar{D})$ the space of infinite sequences $w = (w^1, w^2, \ldots)$ of real-valued functions $w^j = w^j(t, x)$, $j \in \mathbb{N}$, defined and continuous in a domain $\bar{D}$, such that

$$\sup\left\{|w^j| : j \in \mathbb{N}\right\} < \infty$$

equipped with the norm

$$\|w\|_{\mathscr{C}_\mathbb{N}(\bar{D})} := \sup\left\{|w^j|_0 : j \in \mathbb{N}\right\},$$

where $w^j \in C(\bar{D})$, $j \in \mathbb{N}$, and

$$|w^j|_0 := \sup\left\{|w^j(t, x)| : (t, x) \in \bar{D}\right\} < \infty$$

is the norm in the space $C(\bar{D})$ of all functions continuous in a domain $\bar{D}$.

The partial order "$\leq$" in the space $\mathscr{C}_\mathbb{N}(\bar{D})$ is defined by means of the positive cone

$$\mathscr{C}^+_\mathbb{N}(\bar{D}) := \{w \colon w = \{w^j\}_{j\in\mathbb{N}} \in \mathscr{C}_\mathbb{N}(\bar{D}),\ w^j(t, x) \geq 0 \quad \text{for } j \in \mathbb{N},\ (t, x) \in \bar{D}\}$$

in the following way

$$u \leq v \Leftrightarrow v - u \in \mathscr{C}^+_\mathbb{N}(\bar{D}).$$

From this it follows that the inequality $u(t,x) \leq v(t,x)$ is to be understood component-wise, i.e., $u^j(t,x) \leq v^j(t,x)$ for all $j \in \mathbb{N}$. Inequality $u \leq v$ is to be understood both component-wise and point-wise, i.e., $u^j(t,x) \leq v^j(t,x)$ for arbitrary $(t,x) \in \bar{D}$ and all $j \in \mathbb{N}$.

We introduce the space $\mathscr{C}_{N,0}(\bar{D})$, consisting of those infinite sequences in $\mathscr{C}_{\mathbb{N}}(\bar{D})$ which have the following form $w_{N,0} = (w_N^1, w_N^2, \ldots, w_N^N, 0, 0, \ldots)$ where $w_N^j \in \mathbb{R}$ for $j = 1, 2, \ldots, N$ and $w_N^j \equiv 0$ for $j = N+1, N+2, \ldots$ such that

$$\max\left\{\left|w_N^j\right|_0 : j = 1, 2, \ldots, N\right\} < \infty$$

equipped with the norm

$$\|w_{N,0}\|_{\mathscr{C}_{N,0}(\bar{D})} = \max\left\{\left|w_N^j\right|_0 : j = 1, 2, \ldots, N\right\}.$$

The space $\mathscr{C}_{N,0}(\bar{D})$ is a subspace of $\mathscr{C}_{\mathbb{N}}(\bar{D})$.

The third space $C_N(\bar{D})$ is the space of finite sequences $w_N = (w_N^1, w_N^2, \ldots, w_N^N)$ of real-valued functions $w_N^j = w_N^j(t,x)$ for $j = 1, 2, \ldots, N$, defined and continuous in a domain $\bar{D}$, such that

$$\max\left\{\left|w^j\right|_0 : j = 1, 2, \ldots, N\right\} < \infty$$

equipped with the norm

$$\|w_N\|_{C_N(\bar{D})} := \max\left\{\left|w^j\right|_0 : j = 1, 2, \ldots, N\right\}.$$

These three spaces are Banach sequence spaces. In these spaces we have to do with the component-wise convergence.

We remark that the partial ordering in the space $\mathscr{C}_{\mathbb{N}}(\bar{D})$ induces a corresponding partial ordering in the subspaces $\mathscr{C}_{N,0}(\bar{D})$ and $C_N(\bar{D})$.

Convention. We adhere to the convention that every infinite sequence

$$w_{N,0} = (w_N^1, \ldots, w_N^j, \ldots, w_N^N, 0, 0, \ldots) \in \mathscr{C}_{N,0}(\bar{D})$$

is treated as finite one

$$w_N = (w_N^1, \ldots, w_N^j, \ldots, w_N^N) \in C_N(\bar{D}),$$

which we will write as

$$w_{N,0} \cong w_N \quad \text{for all } N \in \mathbb{N}.$$

In this sense, the space $\mathscr{C}_{N,0}(\bar{D})$ is identified with the space $C_N(\bar{D})$, which we will write as

$$\mathscr{C}_{N,0}(\bar{D}) \cong C_N(\bar{D}). \tag{1}$$

Finally, in this sense, the space $C_N(\bar{D})$ may be treated as the subspace of the space $\mathscr{C}_{\mathbb{N}}(\bar{D})$.

Let us consider weakly coupled[a] infinite countable systems of semilinear[b] parabolic differential-functional equations of the reaction-diffusion-convection type of the form[c]

$$\mathcal{F}^j[z^j](t,x) := \mathcal{D}_t z^j(t,x) - \mathcal{L}^j[z^j](t,x) = f^j(t,x,z(t,x),z) \tag{2}$$

for $j \in \mathbb{N}$, where

$$\mathcal{L}^j := \sum_{i,k=1}^{m} a_{ik}^j(t,x)\mathcal{D}_{x_i x_k}^2 - \sum_{i=1}^{m} b_i^j(t,x)\mathcal{D}_{x_i}$$

are diffusion-convection operators and $x = (x_1, \ldots, x_m)$, $(t,x) \in (0,T] \times G := D$, $0 < T < \infty$, where T can be arbitrarily large, $G \subset \mathbb{R}^m$ and G is an open and bounded domain, whose boundary ∂G is an $(m-1)$-dimensional surface of a class $C^{2+\alpha}$ $(0 < \alpha < 1)$, $S_0 := \{(t,x) : t = 0, x \in \bar{G}\}$, $\sigma := [0,T] \times \partial G$ is a lateral surface of a cylindrical domain D, $\Gamma := S_0 \cup \sigma$ is the parabolic boundary of domain D and $\bar{D} := D \cup \Gamma$, $\mathbb{N}$ is the set of natural numbers and N is an arbitrary fixed natural number.

Diagonal operators $\mathcal{F}^j$, $j \in \mathbb{N}$, are uniformly parabolic in $\bar{D}$, z stands for the functions

$$z\colon \mathbb{N} \times \bar{D} \to \mathbb{R}, \quad (j,t,x) \mapsto z(j,t,x) := z^j(t,x),$$

composed of unknown functions $z = \{z^j\}_{j\in\mathbb{N}} := (z^1, z^2, \ldots)$, and f^j, $j \in \mathbb{N}$, are given nonlinear functions

$$f^j\colon\ \Omega := \bar{D} \times \ell^\infty \times \mathscr{C}_{\mathbb{N}}(\bar{D}) \to \mathbb{R}, \quad (t,x,y,s) \mapsto f^j(t,x,y,s), \ \ j \in \mathbb{N}.$$

The right-hand sides f^j of the equations, i.e., the reaction functions (reaction terms) which describe kinetic behavior of considered process, are functionals with respect to the last variable and we assume that they are the Volterra-type.

If we introduce the function $\tilde{f} = \{\tilde{f}^j\}_{j\in\mathbb{N}}$ setting

$$\tilde{f}^j(t,x,z) := f^j(t,x,z(t,x),z), \quad j \in \mathbb{N},$$

where

$$\tilde{f}^j\colon\ \tilde{\Omega} := \bar{D} \times \mathscr{C}_{\mathbb{N}}(\bar{D}) \to \mathbb{R}, \quad (t,x,s) \mapsto \tilde{f}^j(t,x,s), \ \ j \in \mathbb{N},$$

then we will write the equations of system (2) in another form, which may be useful in our further considerations:

$$\mathcal{F}^j[z^j](t,x) = \tilde{f}^j(t,x,z) := \tilde{f}^j(t,x,z^1,z^2,\ldots) \tag{3}$$

for $j \in \mathbb{N}$, $(t,x) \in D$.

[a] This means that every equation contains derivatives of one unknown function only.

[b] Semilinear equations are linear with respect to partial second order special derivatives with coefficients dependent on the variables (t,x) only.

[c] The notation w denotes that w is regarded as an element of the set of admissible functions, while $w(t,x)$ stands for the value of this function at time t and on the point x. However, sometimes, to stress the dependence of function w on the variables t and x, we will write $w = w(t,x)$ and hope that this will not confuse the reader.

For system (2) we will consider the Fourier first initial-boundary value problem: Find the regular (classical) solution of system (2) (or (3)) in $\bar{D}$ fulfilling the initial-boundary condition

$$z(t,x) = \phi(t,x) \quad \text{for } (t,x) \in \Gamma \tag{4}$$

or the homogeneous initial-boundary condition

$$z(t,x) = 0 \quad \text{for } (t,x) \in \Gamma. \tag{5}$$

3. Remarks on the Truncation Method

It is not possible to solve directly infinite countable and uncountable systems of differential equations (cf. [49]). In practice, an infinite system of differential equations is replaced by a finite system of suitably defined differential equations. Such transition from infinite systems of equations to finite ones, known as truncation, may be effected in various ways. One of the ways to describe the truncation process is by assuming that we have to project from an appropriately chosen infinite-dimensional abstract functions space onto its finite-dimensional subspaces.

In the case of infinite countable system, this will be the classical partially ordered infinite-dimensional Banach space of convergent sequences of real-valued functions $\mathscr{C}_{\mathbb{N}}(\bar{D})$ and then $C_N(\bar{D})$ will be its finite-dimensional subspace [18]. In the case of infinite uncountable systems, the studies are carried out in the Banach space L_1 (cf. [15]) which is natural from the physical point of view or in a suitably chosen Hilbert space.[d]

In the truncation method, solutions of the infinite systems of differential equations are defined as the limits when $N \to \infty$ of the sequences of approximations, which are solutions of the finite truncated systems of the first N equations in N unknown functions with the corresponding initial-boundary conditions. However, we observe that we do not need to know the previous approximations to determine the next approximations.

In the second step of the truncation method one has to prove that finite truncated systems of considered equations have the solutions in the suitable space.

It should be emphasized here that there are many existence theorems for finite systems of ordinary differential equations or semilinear parabolic differential-functional equations of the reaction-diffusion type. These existence theorems have been proven by means of the monotone iterative method (method of lower and upper solutions) [5, 6, 8, 22], the topological fixed point method [6, 45–48], the theory of continuous semigroups of linear operators and evolution systems techniques [35], as well as the finite difference method [34], just to mention a few.

If the truncated systems are solved by the method of continuous semigroups of linear operators, then we must recast the considered system with the initial condition

[d] Hilbert spaces plays an important role in theoretical physics, including in particular non-relativistic quantum mechanics (cf. [32]).

as an abstract Cauchy problem. Next, by using the truncation argument and applying a certain fixed point theorem, we prove that, under some assumptions, the abstract Cauchy problem has a unique solution.

If the truncated systems of parabolic equations are solved by means of the finite difference method, then we will have three basic computational monotone iterative schemes to choose from: the Picard iteration, a modified version of Jacobi iteration, or the Gauss-Seidel method. If the initial iteration is always a pair of known coupled lower and upper solutions of the problem considered, then the sequence generated by the Picard iteration converges faster than the sequence generated by the Gauss-Seidel iteration, which, in turn, converges faster than the sequence generated by the Jacobi iteration (cf. [34]).

We note that the finite difference method is not only one of the simplest methods used in numerical analysis, but also an important analytical method of proving existing theorems in the field of partial differential equations. It is highly advantageous to choose a finite difference method to prove the existence and uniqueness of any truncated system for the problem considered, and we thus arrive at numerically proven constructive theorems on existence. Each of these solutions is an approximation of a solution of this problem and may forthwith be calculated and plotted or tabulated.

Applying the method of lower and upper solutions requires assuming the monotonicity and the Lipschitz condition of the reaction functions $f^j = f^j(t,x,y,s)$, $j \in \mathbb{N}$ with respect to the function and functional arguments y and s, respectively. The right-hand side of the Lipschitz condition is used to ensure the uniqueness of the solution and the left-hand side condition is necessary to ensure the existence of the solution. We also assume the existence of an ordered pair of a lower and upper solutions of the problem considered.

The third step of the truncation method will be in finding an additionally sufficient condition, which, while adding to the previous assumptions, will guarantee the existence of a regular limit function for approximating sequences of solutions of the truncated systems, where this function — in line with the definition adopted — will be a solution to our problem. Condition requiring the reaction functions to be bounded by the terms of a convergent sequence of reals turns out to be such a sufficient condition. Therefore, we will assume the following sufficient condition:

B: *Suppose that the reaction functions* $f^j = f^j(t,x,y,s)$, $j \in \mathbb{N}$, *are continuous and there exists a sequence* $\{q_j\}_{j\in\mathbb{N}}$ *of non-negative real numbers* $q_j \geq 0$ *for* $j \in \mathbb{N}$ *such that* $|f^j(t,x,y,s)| \leq q_j$ *for* $j \in \mathbb{N}$, *where* $\lim_{j\to\infty} q_j = 0$.

This condition plays a crucial role in proving the existence theorem. Similar conditions appear in papers [36–39, 41, 42, 50, 55], where infinite countable systems of ordinary differential equations are studied (cf. also [8]). Moreover, such an assumption makes sense and is physically justified.

The uniqueness of the solution of the problem considered is guaranteed by the Lipschitz condition (precisely by the right-hand side Lipschitz condition) and follows from Szarski's uniqueness criterion (see [44], cf. also [16, 19, 20]).

4. Truncation Methods for Infinite Countable Systems of Differential Equations

In this section, we present the truncation method for this problem in partially ordered Banach sequence spaces $\mathscr{C}_{\mathbb{N}}(\bar{D})$.

In the truncation method, a solution z of infinite countable system (3) is defined as the limit when $N \to \infty$ of the sequences of approximations $\{z_N\}_{N=1,2,\ldots}$, where $z_N = (z_N^1, z_N^2, \ldots, z_N^N)$ are defined as solutions of finite systems of the first N equations of system (3) in N unknown functions (i.e., truncated system) of the form

$$\begin{aligned}\mathcal{F}^j[z_N^j](t,x) &= \tilde{f}^j(t,x,z_N^1,\ldots,z_N^j,\ldots,z_N^N,\psi^{N+1},\psi^{N+2},\ldots) \\ &:= \tilde{f}^j_{N,\psi}(t,x,z_N) \quad \text{for } j=1,2,\ldots,N, \quad (t,x)\in D,\end{aligned} \tag{6}$$

with the corresponding initial-boundary conditions of the form

$$z_N^j(t,x) = \phi^j(t,x) \quad \text{for } j=1,2,\ldots,N, \quad (t,x)\in\Gamma. \tag{7}$$

The remaining terms $z_N^{N+1}, z_N^{N+2}, \ldots$ of the approximation sequences $\{z_N\}_{N=1,2,\ldots}$ are defined as follows:

$$z_N^j(t,x) := \psi^j(t,x) \quad \text{for } j=N+1,N+2,\ldots, \quad (t,x)\in\bar{D} \tag{8}$$

where the function $\psi = \{\psi^j\}_{j\in\mathbb{N}}$, $\psi^j = \psi^j(t,x)$, defined for $(t,x)\in\bar{D}$, and satisfying initial-boundary condition (7)

$$\psi(t,x) = \phi(t,x) \quad \text{for } (t,x)\in\Gamma,$$

will be determined later on.

Amann [2] noticed that in the case of solving countable systems of equations as the discrete coagulation-fragmentation models with diffusion, the technique used in practically all papers is the natural one: it starts with a study of finite systems obtained by truncation to the first N equations, followed by passing to the limit as $N \to \infty$.

We note that to solve infinite countable systems of ordinary differential equations, partial differential equations of parabolic type and integro-differential equations, numerous authors have applied the truncation method (see [4, 12, 21, 23, 24, 28–31, 33, 36–39, 41, 42, 50–55).

Let us consider the infinite-dimensional Banach sequence space $\mathscr{C}_{\mathbb{N}}(\bar{D})$ and this subspace $\mathscr{C}_{N,0}(\bar{D})$.

We define the operator ρ_N as follows:

$$\begin{gathered}\rho_N :\ \mathscr{C}_{\mathbb{N}}(\bar{D}) \to \mathscr{C}_{N,0}(\bar{D}), \\ (z^1, z^2, \ldots) = z \mapsto \rho_N[z] := z_{N,0} = (z^1,\ldots,z^j,\ldots,z^N,0,0,\ldots)\end{gathered} \tag{9}$$

for all $z\in\mathscr{C}_{\mathbb{N}}(\bar{D})$ and an arbitrary $N\in\mathbb{N}$.

This means that

$$\rho_N[z] := z_{N,0} = \begin{cases} z^j & \text{when } 1 \le j \le N, \\ 0 & \text{when } j > N \end{cases} \tag{10}$$

for all $z \in \mathscr{C}_{\mathbb{N}}(\bar{D})$ and an arbitrary $N \in \mathbb{N}$.

It is easy to show that

$$\rho_N^2[z] = \rho_N[\rho_N[z]] = \rho_N[z_{N,0}] = z_{N,0} = \rho_N[z]$$

for all $z \in \mathscr{C}_{\mathbb{N}}(\bar{D})$ and $N \in \mathbb{N}$. Therefore, ρ_N is the projection operator.

We also define the projection operator q_N by:

$$\begin{gathered} q_N : \mathscr{C}_{\mathbb{N}}(\bar{D}) \to C_N(\bar{D}), \\ (z^1, z^2, \ldots) = z \mapsto q_N[z] := z_N = (z^1, z^2, \ldots, z^N) \end{gathered} \tag{11}$$

for all $z \in \mathscr{C}_{\mathbb{N}}(\bar{D})$ and an arbitrary $N \in \mathbb{N}$.

By convention (1), we treat that

$$\rho_N[z] \cong q_N[z] \tag{12}$$

for all $z \in \mathscr{C}_{\mathbb{N}}(\bar{D})$ and $N \in \mathbb{N}$.

As an example, we consider the infinite countable system of equations of form (2) with initial-boundary condition (4), i.e.,

$$\begin{cases} \mathcal{F}^j[z^j](t,x) = f^j(t,x,z(t,x),z) := \tilde{f}^j(t,x,z) & \text{for } (t,x) \in D, \\ z^j(t,x) = \phi^j(t,x) \quad \text{for } (t,x) \in \Gamma, \end{cases} \tag{13}$$

for $j \in \mathbb{N}$, in the infinite-dimensional Banach sequence space $\mathscr{C}_{\mathbb{N}}(\bar{D})$, where the reaction functions are given as [51], i.e., these have the special form:

$$\begin{aligned} \tilde{f}^1(t,x,z) &:= -z^1(t,x) \sum_{k=1}^{\infty} a_k^1 z^k(t,x) + \sum_{k=1}^{\infty} b_k^1 z^{1+k}(t,x), \\ \tilde{f}^j(t,x,z) &:= \frac{1}{2} \sum_{k=1}^{j-1} a_k^{j-k} z^{j-k}(t,x) z^k(t,x) \\ &\quad - z^j(t,x) \sum_{k=1}^{\infty} a_k^j z^k(t,x) \\ &\quad + \sum_{k=1}^{\infty} b_k^j z^{j+k}(t,x) - \frac{1}{2} z^j(t,x) \sum_{k=1}^{j-1} b_k^{j-k} \quad \text{for } j = 2, 3 \ldots, \end{aligned} \tag{14}$$

where the coagulation rates a_k^j and the fragmentation rates b_k^j are non-negative constants such that $a_k^j = a_j^k$ and $b_k^j = b_j^k$.

This system may be treated as the discrete mathematical model of the coagulation-fragmentation processes with diffusion.

If to this problem we use the projection operator q_N defined by (11) then we obtain the finite truncated problem considered in the subspace $C_N(\bar{D})$

$$\begin{cases} \mathcal{F}^j[z_N^j](t,x) = \tilde{f}_N^j(t,x,z_N) & \text{for } (t,x) \in D, \\ z_N^j(t,x) = \phi^j(t,x) & \text{for } (t,x) \in \Gamma \end{cases} \tag{15}$$

for $j = 1, 2, \ldots, N$ and an arbitrary $N \in \mathbb{N}$, where $z_N = (z_N^1, \ldots, z_N^j, \ldots, z_N^N) \in C_N(\bar{D})$ and

$$\begin{aligned} \tilde{f}_N^1(t,x,z_N) &:= -z_N^1(t,x) \sum_{k=1}^{N} a_k^1 z_N^k(t,x) + \sum_{k=1}^{N} b_k^1 z_N^{1+k}(t,x), \\ \tilde{f}_N^j(t,x,z_N) &= \frac{1}{2} \sum_{k=1}^{j-1} a_k^{j-k} z_N^{j-k}(t,x) z_N^k(t,x) - z_N^j(t,x) \sum_{k=1}^{N} a_k^j z_N^k(t,x) \\ &\quad + \sum_{k=1}^{N} b_k^j z_N^{j+k}(t,x) \\ &\quad - \frac{1}{2} z_N^j(t,x) \sum_{k=1}^{j-1} b_k^{j-k} \quad \text{for } j = 2, 3, \ldots, N \end{aligned} \tag{16}$$

with the corresponding initial-boundary condition (4).

If we will truncate this system by acceptance

$$a_k^j \equiv 0 \quad \text{and} \quad b_k^j \equiv 0 \quad \text{for } j > N \quad \text{or} \quad k > N, \tag{17}$$

then we obtain the truncated system (15), (16).

This means that the truncation of the infinite system to the system of the first N equations of the infinite system in N unknown functions, may be treated as a projection of the infinite countable system considered in the infinite-dimensional Banach sequence space $\mathscr{C}_{\mathbb{N}}(\bar{D})$ onto its finite N-dimensional subspace $C_N(\bar{D})$.

5. Truncation of Infinite Uncountable Systems of Differential Equations

The truncation method is also the fundamental approximation method of studying solvability of infinite uncountable systems of ordinary differential equations, and parabolic differential equation of the reaction-diffusion type, as well as integro-differential and differential-functional equations (cf. [23, 24, 28–31]). A finite truncated system may be obtained from an uncountably infinite system with use of a projection.

McLaughlin *et al.* [28–31], Lamb [23] and Laurençot [24] used such an approach in dealing with a continuous mathematical model for the dynamics of cluster growth under the combined effect of two processes: coagulation and fragmentation. These processes are modeled by the continuous coagulation and multiple-fragmentation

equations, and investigated by means of the theory of semigroup of linear operators, evolution systems and fixed point mapping techniques.

McLaughlin *et al.* [28] consider the pure multiple-fragmentation process for time independent kernels $\gamma = \gamma(\lambda, y)$, which is described by the following autonomous integro-differential linear equation

$$\frac{\partial}{\partial t} z(t,\lambda) = \int_\lambda^\infty \gamma(y,\lambda)\, z(t,y) dy \\ - z(t,\lambda) \int_0^\lambda \frac{y}{\lambda} \gamma(\lambda, y) dy \quad \text{for } \lambda > 0, t > 0 \tag{18}$$

with initial condition

$$z(0,\lambda) = f(\lambda) \quad \text{for } \lambda > 0. \tag{19}$$

This problem is dependent on an additional real positive parameter[e] $\lambda \in [\lambda_0, \lambda_1] \subset \mathbb{R}^+$ and

$$z\colon\ (0,T] \times [\lambda_0, \lambda_1] \to \mathbb{R}^+, \quad (t,\lambda) \mapsto z(t,\lambda). \tag{20}$$

We remark that Eq. (18) can be treated as a system of many uncountable integro-differential equations.

To apply the theory of semigroups of linear operators we must recast Eq. (18) with the initial condition (19) as a linear abstract Cauchy problem:

$$\begin{cases} \dfrac{d}{dt} \tilde{z}(t) = A[\tilde{z}](t) \quad \text{for } t > 0 \\ \tilde{z}(0) = f \end{cases} \tag{21}$$

where $\tilde{z} = \tilde{z}(t)$ is the suitable defined function.

For this purpose we consider the Banach space $\mathcal{X}$ of type L (see [15]), precisely the $L_{1,-1}$ space of equivalence classes of measurable, real-valued functions ϕ such that

$$\int_0^\infty \lambda |\phi(\lambda)| d\lambda < \infty$$

equipped with the weighted norm

$$\|\phi\|_{1,-1} := \int_0^\infty \lambda |\phi(\lambda)| d\lambda.$$

Finally, the pure fragmentation Eq. (18) with initial condition (19) can be recast as the abstract Cauchy problem

$$\begin{cases} \dfrac{d}{dt} z(t) = A[z](t) \quad \text{for } t > 0, \\ z(0) = f \end{cases} \tag{22}$$

where the linear operator

$$A : L_{1,-1} \supseteq D_{\max}(A) \to L_{1,-1}$$

[e] Each cluster is identified by its size which is assumed to be a positive real number. The volume is used as the characteristic size. Therefore, by the size of a cluster we are referring to its volume.

is defined on its maximal domain $D_{\max}(A)$ by

$$A[\phi](\lambda) := \int_{\lambda}^{\infty} \gamma(y,\lambda)\phi(y)dy \\ - \phi(\lambda)\int_{0}^{\lambda} \frac{y}{\lambda}\gamma(\lambda,y)dy \quad \text{for all } \phi \in D_{\max}(A). \tag{23}$$

In this problem the conservation of mass that is the identity

$$\int_0^\infty \lambda\, z(t,\lambda)d\lambda = \int_0^\infty \lambda\, f(\lambda)d\lambda \quad \text{for all } t \geq 0$$

which implies that a natural space to work with is a Banach space $\mathcal{X}$ of the type L.

If we consider the subspace L_N of the Banach space $L_{1,-1}$ consisting with functions vanishing when $\lambda \geq N$, $N \in \mathbb{N}$, i.e.,

$$L_N := \{\phi \in L_{1,-1} : \phi = 0 \quad \text{on } [N,\infty)\}, \tag{24}$$

then we define the projection operator P_N on the space $L_{1,-1}$ onto the subspace L_N as follows

$$P_N : L_{1,-1} \to L_N, \quad z \mapsto P_N[z] := z_N$$

for all $z \in L_{1,-1}$ and an arbitrary $N \in \mathbb{N}$.

This means that

$$P_N[z](\lambda) := \begin{cases} z(\lambda) & \text{when } 0 < \lambda < N, \\ 0 & \text{when } \lambda \geq N, \end{cases} \tag{25}$$

for all $z \in L_{1,-1}$ and an arbitrary $N \in \mathbb{N}$.

If we use the projection operator P_N defined by (25) to the abstract Cauchy problem (22), then we obtain the truncated problem

$$\begin{cases} \dfrac{d}{dt}z(t) = A_N[z](t) & \text{for } t > 0, \\ z(0) = f \end{cases} \tag{26}$$

where $A_N = AP_N$. The operators A_N are defined as follows

$$A_N[\phi](\lambda) := \begin{cases} \displaystyle\int_\lambda^N \gamma(y,\lambda)\phi(y)dy - \phi(\lambda)\int_0^\lambda \frac{y}{\lambda}\gamma(\lambda,y)dy & \text{for } 0 < \lambda < N, \\ 0 \quad \text{for } \lambda \geq N, \end{cases}$$

for all $\phi \in L_{1,-1}$ and an arbitrary $N \in \mathbb{N}$.

Applying the classical methods of the theory of continuous semigroups of linear operators and evolution system techniques (see [28,35]) and the Banach fixed point theorem for contraction mappings, we prove the existence and uniqueness of a weak solution for the abstract Cauchy problem (22).

6. Relation Between Continuous and Discrete Infinite-Dimensional Models

Amann [2] consider the initial value problem for semilinear reaction-diffusion equation with convection of the form

$$\frac{\partial z}{\partial t} - A(t,x,\lambda)z = f(t,x,z(t,x,\lambda),\lambda) \tag{27}$$

for $t > 0$, $x \in \mathbb{R}^m$, $m = 1, 2$, or 3, with initial condition

$$z(0,x,\lambda) = z_0(x,\lambda) \quad \text{for } x \in \mathbb{R}^m, \tag{28}$$

which is a continuous model of the coagulation-fragmentation processes with diffusion.

In order to present the difficulties and obtain results in a precise and relatively comprehensive manner, we quote Amann (see [2, pp. 340, 344]): "This problem depend on an additional real parameter λ, the volume. The diffusion-convection operator A is uniformly elliptic, the reaction function (reaction term) f describes kinetic behavior of the process and $z = z(t,x,\lambda)$ is the particle-size distribution function.

It is easy that the above equation can be viewed as a coupled system of infinite uncountable many reaction-diffusion equations.

In the event of the continuous model, the right-hand sides of equations include terms of the form:

$$\mathcal{J} = \int_G \int_{\lambda_0}^{\lambda_1} z(t,x,\lambda)dx\, d\lambda \tag{29}$$

which is the total number of particles with volumes belonging to the interval $[\lambda_0, \lambda_1] \subset \mathbb{R}^+$ and being at time t contained in the space region $G \subset \mathbb{R}^m$. The measure $d\lambda$ is either Lebesgue's measure on $\mathbb{R}^+$ or the counting measure on $\mathbb{N} := \{1, 2, 3, \ldots\}$. In the latter case only clusters whose sizes are integer multiples of an "elementary unit" can occur. In this case, all integrals with respect to $d\lambda$ reduce to sums, so that Eq. (29) takes the form:

$$\mathscr{S} = \int_G \sum_{\lambda=\lambda_0}^{\lambda_1} z(t,x,\lambda)dx. \tag{30}$$

This situation corresponds to the discrete model of the coagulation-fragmentation processes considered."

Next, Amann [2, p. 345], consider problems (27), (28) as a single semilinear evolution equation

$$\frac{dz}{dt} - A(t)z = f(t,z) \quad \text{for } t > 0, \tag{31}$$

with initial condition

$$z(0) = z_0, \tag{32}$$

where $z = z(t, x)$ is a Banach-space-valued function of $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^m$. This means that Eq. (31) may be interpreted as a vector-valued evolution equation which we are able to handle using recent Fourier multiplier theorems for operator-valued symbols and Banach-space-valued distributions obtained in an earlier paper [1]. It should be noted that in that paper, the author used highly abstract notions and methods. The main result is the theorem on existence and uniqueness of solution of continuous coagulation-fragmentation equation with diffusion in the class of volume preserving solutions. An advantage of this approach is the feasibility of the simultaneous consideration of both the continuous and discrete models of coagulation-fragmentation processes with diffusion.

7. Problems and Conclusions

1. What is the relationship between solutions of the infinite uncountable system of equations serving as a continuous model of process considered and solutions of the infinite countable system of equations serving as a discrete model of this process?

Let us consider two infinite systems of differential equations serving as a continuous model (infinite uncountable system) and a discrete model (infinite countable system), respectively, of the same process. Let us take into consideration expressions of form (29), (30) appearing on the right-hand sides of the respective equations of systems for a special case where G is a bounded domain $G \subset \mathbb{R}^m$, and

$$z\colon\ \mathbb{R}^+ \times G \times [1, \infty) \ni (t, x, \lambda) \mapsto z(t, x, \lambda) \in \mathbb{R}^+$$

is a given monotone decreasing function with respect to the variable λ.

If

$$a_n(t, x) := z(t, x, n) \quad \text{for } (t, x) \in D := \mathbb{R}^+ \times G,$$

and

$$\begin{aligned} \mathscr{S} &:= \int_G \sum_{n=1}^{\infty} a_n(t, x)dx, \\ \mathcal{J} &:= \int_G \int_1^{\infty} z(t, x, \lambda)dx\, d\lambda \quad \text{for } t \in \mathbb{R}^+, \end{aligned} \tag{33}$$

then the sequence of functions

$$\{S_N - I_N\} = \{S_N(t, x) - I_N(t, x)\} \tag{34}$$

given by

$$\begin{aligned} S_N &= S_N(t, x) := \sum_{n=1}^{N} a_n(t, x), \\ I_N &= I_N(t, x) := \int_1^{N} z(t, x, \lambda)d\lambda \quad \text{for } (t, x) \in D \end{aligned} \tag{35}$$

is always convergent in the point-wise sense, and

$$0 \le \lim_{N\to\infty} [S_N(t,x) - I_N(t,x)] \le a_1(t,x) \quad \text{for } (t,x) \in D. \tag{36}$$

In fact,[f] the sequence (34) is decreasing because

$$\begin{aligned}(S_{N-1} - I_{N-1}) - (S_N - I_N) &= (I_N - I_{N-1}) - (S_N - S_{N-1}) \\ &= \int_{N-1}^{N} z(t,x,\lambda)d\lambda - a_N > 0,\end{aligned}$$

and is bounded, because

$$a_N > 0 \quad \text{and} \quad a_N \le S_N - I_N \le a_1.$$

Hence, the sequence (34) converges in the point-wise sense and (36) holds.

Assuming that $a_1 = a_1(t,x)$ is bounded function and $|a_1(t,x)| \le \tilde{a}_1$ in $\bar{D}$ we obtain the following estimate

$$0 \le \lim_{N\to\infty} [S_N(t,x) - I_N(t,x)] \le \tilde{a}_1 \quad \text{for } (t,x) \in \bar{D}. \tag{37}$$

The above-mentioned properties of the sequence $\{S_N(t,x) - I_N(t,x)\}$ may, under appropriate assumptions, extend onto the sequences $\{\mathscr{S}_N(t) - \mathcal{J}_N(t)\}$, where

$$\begin{aligned}\mathscr{S}_N &= \mathscr{S}_N(t) := \int_G \sum_{n=1}^{N} a_n(t,x)dx, \\ \mathcal{J}_N &= \mathcal{J}_N(t) := \int_G \int_1^N z(t,x,\lambda)dx\,d\lambda \quad \text{for } t \in \mathbb{R}^+,\end{aligned} \tag{38}$$

and from these, further still, onto the right-hand sides of the equations. This means that, the right-hand sides of considered systems may be both identical or different. The latter implies that the two infinite systems of equations may both have solutions, but in general different ones.

This means that solutions of infinite countable and infinite uncountable systems of differential equations, which are the discrete and continuous models of this same process, respectively, may be both identical or different.

The above statement is a material result proving that dynamics of the continuous case differs from that of the discrete case.

2. Under what assumptions are both models identical and are the assumptions restrictive?

Unfortunately, at this stage of research, it is difficult to answer this question. Moreover, it should be emphasized that estimate (36) refers to a term of the reaction function only. Does the difference between continuous vs. discrete models also occur when the mathematical problem is linear? These problems will be the subject of further research.

[f] An idea is similar to this given in Ref. [25].

3. A very important problem is the comparison between the infinite countable system and finite truncated systems of these equations.

The comparison between the infinite countable system of parabolic differential equations (in particular, between the solution of initial-boundary value problems) and finite systems of these equations is possible under suitable assumptions for reaction functions.

The basic tools for proving these properties will be comparison theorems and maximum principles for finite and infinite systems of parabolic differential-functional equations, and the Gronwall-Bellman lemma (see [16, 19, 20, 44]). This problem will also be the subject of further research.

4. Concluding our considerations, we see that approximations of the solution $z = z(t, x)$ of the infinite system of differential equations are realized in two steps.

First, we replace the solution $z = z(t, x)$ with the solution $z_N = z_N(t, x)$ of the finite truncated systems where N is such that

$$\|z - z_N\| \leq \frac{\varepsilon}{2} \quad \text{(truncation error)}.$$

Next, we apply a numerical method (e.g., finite difference method) to finite systems to compute numerical approximations $z^h_{N,numer} = z^h_{N,numer}(t, x)$ such that

$$\|z_N - z^h_{N,numer}\| \leq \frac{\varepsilon}{2} \quad \text{(numerical method error)}.$$

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