Mathematical Entertainments

David Gale*

This column is interested in publishing mathematical material which satisfies the following criteria, among others:

1. It should not require technical expertise in any specialized area of mathematics.

2. The topics treated should when possible be comprehensible not only to professional mathematicians but also to reasonably knowledgeable and interested nonmathematicians.

We welcome, encourage and frequently publish contributions from readers. Contributors who wish an acknowledgement of submission should enclose a self-addressed postcard.

Our guest columnist for this issue is Solomon W. Golomb. It has seemed to us fitting to dedicate the column to the memory of Raphael Robinson. As mentioned in an earlier issue, Raphael contributed more material to this column over the past four years than any other outside person. The present column seems particularly appropriate for this dedication since it deals with the subjects of tiling and decidability, areas to which Raphael made fundamental pioneering contributions.

Tiling Rectangles with Polyominoes

Solomon W. Golomb

In 1900, when the great German mathematician David Hilbert, in an address to the International Congress of Mathematicians assembled in Paris, laid out his agenda for twentieth-century mathematics, his "most wanted list" of twenty-three unsolved problems included, as number 18, a question concerning the ways in which n-dimensional Euclidean space (including $n = 2$ and $n = 3$) can be "tiled" or "packed" with congruent copies of a single geometric figure. Specifically, he asked:

> #1. "Is there in n-dimensional Euclidean space . . . only a finite number of different kinds of groups of motions with a [compact] fundamental region?"
> #2. "Whether polyhedra also exist which do not appear as fundamental regions of groups of motions, by means of which nevertheless by a suitable juxtaposition of congruent copies a complete filling up of all [Euclidean] space is possible?"

The groups of motions (#1) were identified by L. Bieberbach, but examples answering #2 in the affirmative were found in 3 dimensions (K. Reinhardt, 1928) and in 2 dimensions (Heesch, 1935). Simpler, related, and more general examples were subsequently found by R.M. Robinson, S.K. Stein, *et al.* An example related to Heesch's is shown in Figure 1. (Smaller examples exist, but it is somewhat harder to prove they have the required property.) It is not hard to show that the only way this figure can tile the plane is as shown in Figure 2. However, it is not a "fundamental domain," for note that the unique motion which carries A onto B combines a reflection about the dotted line with an upward translation by 2 units, but the image of B under this motion is not a tile.

Hilbert never anticipated Gödel's Incompleteness Theorem, let alone the possibility that some of his problems would be proved "undecidable." Just as his tenth problem, involving finding all integer solutions of given ("diophantine") equations was shown (by Julia Robinson, Ju. V. Matijasevic, *et al.*) to be computationally undecidable, so too the general question of whether it is possible to tile the plane with congruent copies of a given finite set of tiles was proved undecidable by Hao Wang. (Wang showed the equivalence of the tiling problem to the "halting problem" for Turing machines, a standard undecidable problem, on the one hand, and to the undecidability of the class of logical formulas containing three quantifiers, the so-called AEA-formulas, on the other.) R. Berger and other students of Wang extended his results, e.g., to showing that the question of whether it is possible to tile the plane (or a smaller region, such as a rectangle) using congruent copies of a

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single geometric figure is also computationally undecidable.

This article is an attempt to summarize what is known about a particular special case of the tiling problem in two dimensions.

The only tiles we shall consider are *polyominoes*, where an "*n*-omino" is any connected figure obtained by taking n identical unit squares, and connecting them along common edges. Thus, except for orientation, there is only one *monomino* (the unit square itself) and one *domino* (the eponymous ancestor of this entire clan). There are two distinct *trominoes*, five different *tetrominoes*, twelve *pentominoes*, thirty-five *hexominoes*, etc. The simpler ones of these are shown in Figure 3. (Note that we do not regard mirror images ("reflections") as two distinct shapes.)

A typical problem in polyomino theory is to determine which polyominoes have the property that un-

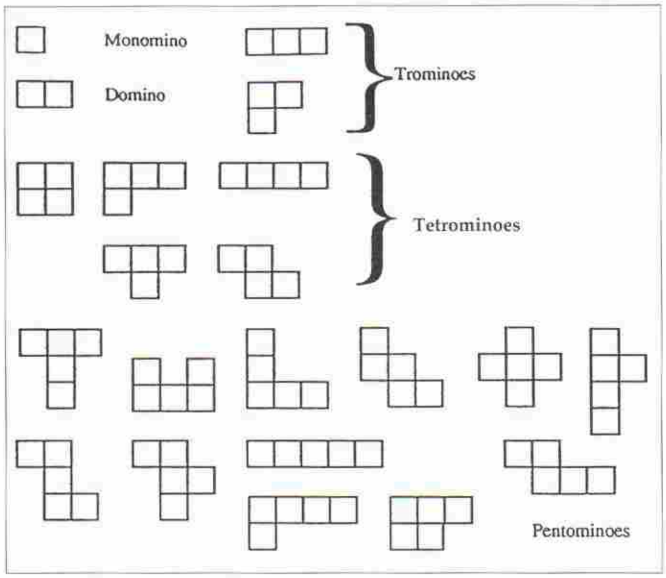

Figure 3. The Simpler Polyominoes.

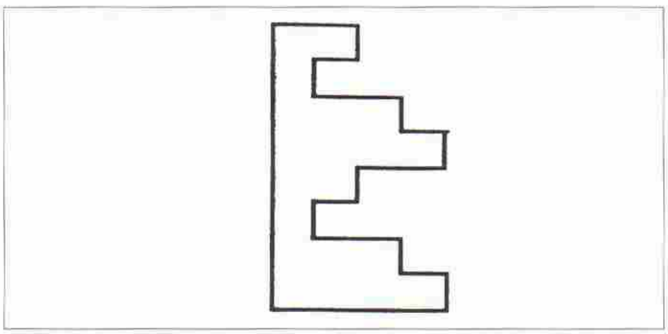

Figure 1. A figure which tiles the plane, but is not a "fundamental region".

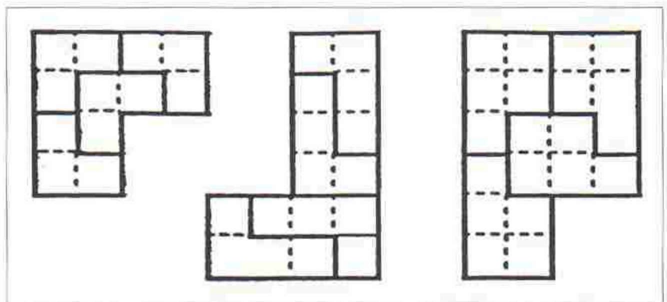

Figure 4. Three "rep-tiles": a tromino, a tetromino, and a pentomino.

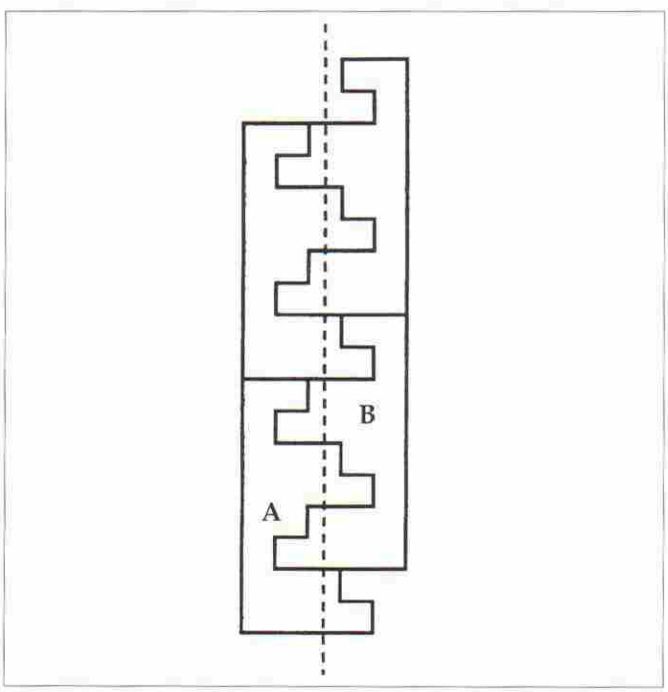

Figure 2. Tiling the plane with the shape in Figure 1.

limited copies of a specific one will tile the entire plane, or will tile a single quadrant of the plane, or will tile an infinite strip bounded by two parallel straight lines, etc. (Figure 2 shows how to tile a strip, hence the plane, with a particular polyomino.) Restrictions can be placed on which, if any, rotations and/or reflections of the basic shape may be used in the tiling. Tilings may be studied in which two, or three, or some other number of different tile shapes are allowed in the tiling. One may ask about those shapes such that several identical copies may be assembled to make an enlarged scale model, or *replica*, of the original shape, as in Figure 4. (Many years ago, in 1962, I coined the term *rep-tiles* for these shapes.)

All of these questions have been studied. However, this article will focus primarily on the question of which polyomino shapes have the property that some finite number of copies of the basic shape, allowing all rotations and reflections, can be assembled to form a rectangle. The problem is challenging because no *a priori* limit can be established, given an arbitrary n-omino, on the minimum number of copies which *may* be required to form a rectangle. No explicit expression in n, such as

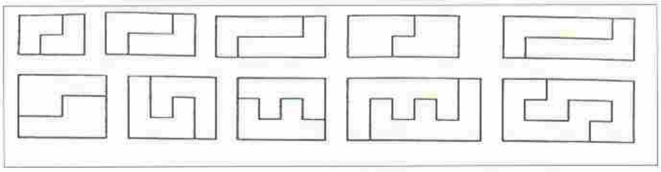

Figure 5. Some polyominoes of order 2.

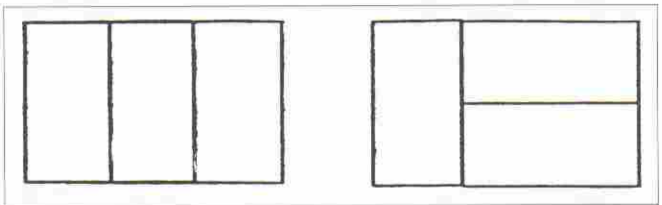

Figure 6. How three identical rectangles can form a rectangle.

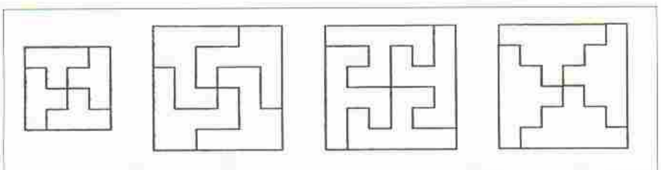

Figure 7. Polyominoes of order 4 under 90° rotation.

n^{n^n}, grows fast enough to guarantee that when all possible arrangements of that many copies of the basic shape have been looked at, and no rectangle has been found, then no rectangle involving yet more copies will exist. This follows from the result, mentioned earlier, that the *general* question of whether an *arbitrary* polyomino will tile a rectangle is "computationally undecidable." Fortunately, in any *specific* case, there is a high likelihood (though no certainty) of answering the question. But there can be no cookbook recipe, no handbook procedure, which can be routinely applied to indicate whether a given polyomino shape will tile some (possibly huge) rectangle. And it is this lack of a general decision procedure which makes this study so interesting and challenging.

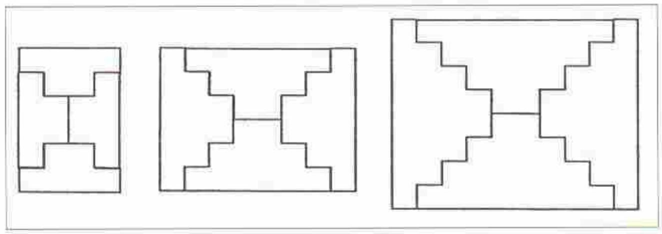

Figure 8. Polyominoes of order 4 under rectangular symmetry.

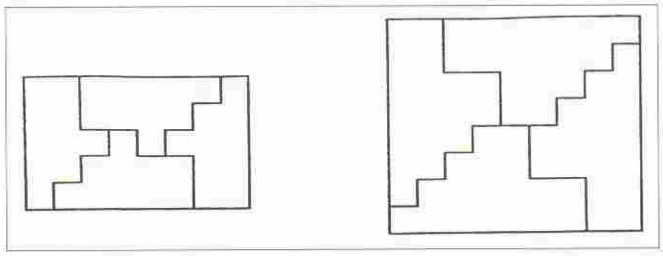

Figure 9. Another order-4 construction by Klarner.

In 1968, David A. Klarner defined the *order* of a polyomino P as the minimum number of congruent copies of P which can be assembled (allowing translation, rotation, and reflection) to form a rectangle. For those polyominoes which will not tile any rectangle, the order is undefined. A polyomino has order 1 if and only if it is itself a rectangle.

A polyomino has order 2 if and only if it is "half a rectangle", since two identical copies of it must form a rectangle. In practice, this means that the two copies will be 180° rotations of each other when forming a rectangle. Some examples are shown in Figure 5.

There are no polyominoes of order 3. (This has been proved by Ian Stewart of the University of Warwick, in

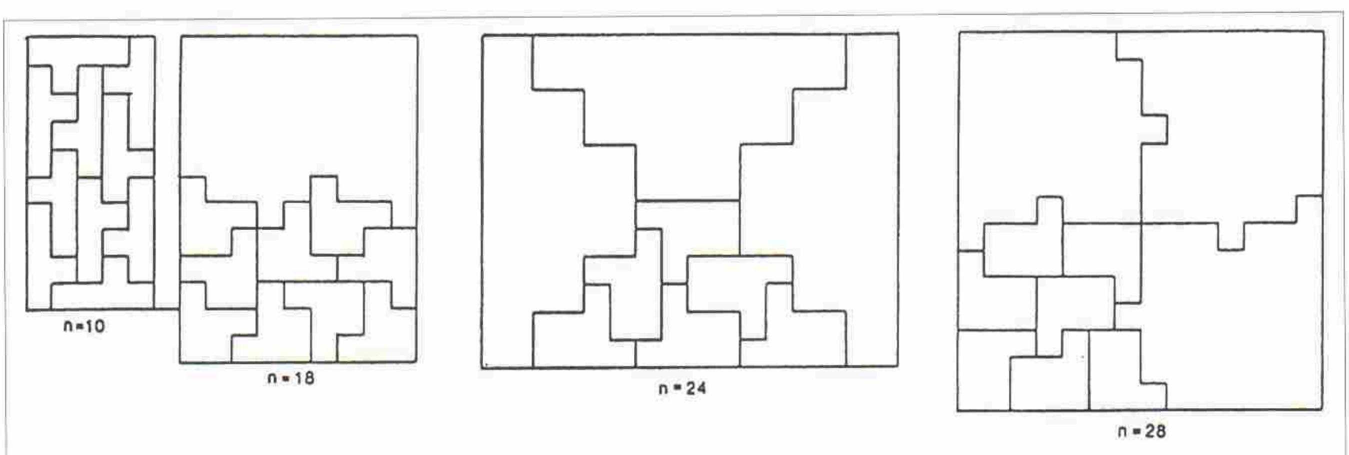

Figure 10. Four "sporadic" polyominoes, of orders 10, 18, 24, and 28, respectively.

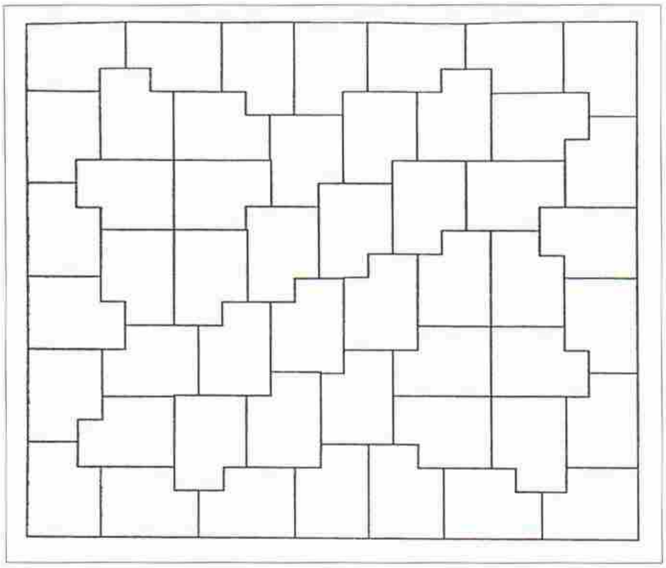

Figure 11. An 11-omino of order 50

England.) In fact, the only way any rectangle can be divided up into three identical copies of a "well-behaved" geometric figure is to partition it into three *rectangles* (see Figure 6), and by definition a rectangle has order 1.

There are various ways in which four identical polyominoes can be combined to form a rectangle. One way, illustrated in Figure 7, is to have four 90° rotations of a single shape forming a square.

Another way to combine four identical shapes to form a rectangle uses the fourfold symmetry of the rectangle itself: left-right, up-down, and 180°-rotational symmetry. Some examples of this appear in Figure 8.

There are also more complicated order-4 patterns which were found by Klarner, two of which are illustrated in Figure 9.

Beyond order 4, there is a systematic construction found by Golomb in 1985 which gives examples of order $4s$ for every positive integer s; and eleven isolated examples of small polyominoes with respective orders 10, 18, 24, 28, 50, 76, 92, 96, 138, 192, and 312 are known.

Figure 10 shows the isolated examples of order 10 (Golomb, 1966), and orders 18, 24, and 28 (Klarner, 1969).

Figure 11 shows the example of order 50, found by William Rex Marshall of Dunedin, New Zealand, in 1990.

Figure 12 shows the examples of orders 76 and 92, both found by Karl A. Dahlke in 1987. I had mentioned these two problems in my talk at the Strens Memorial Conference on Recreational Mathematics (Calgary, 1986), and Ivars Peterson included them in *Science News* in his article about the Strens Conference. When Dahlke sent me solutions which he said he found using only a personal computer, I notified Peterson, who interviewed Dahlke and learned that he is totally blind. I later heard that these two tilings had actually been discovered earlier, by T.W. Marlow in 1985.

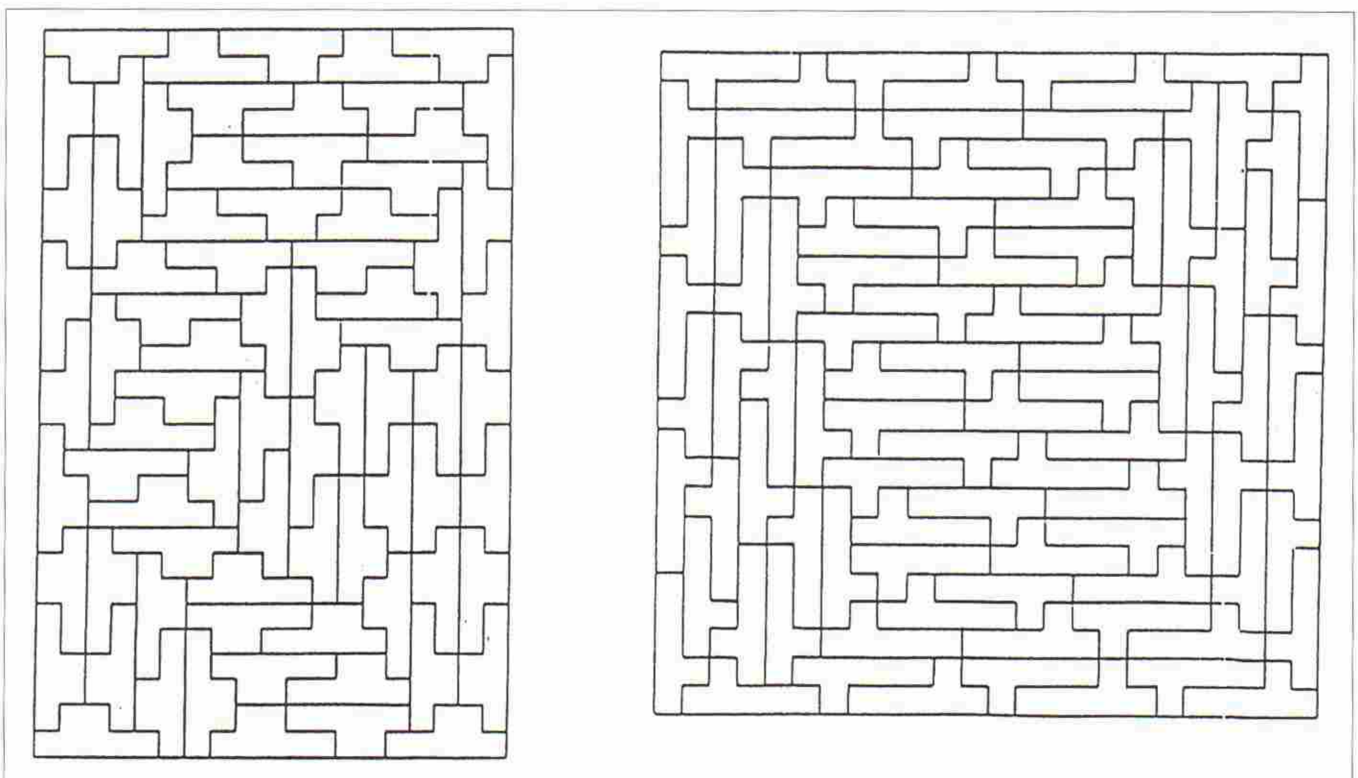

Figure 12. A heptomino of order 76 and a hexomino of order 92.

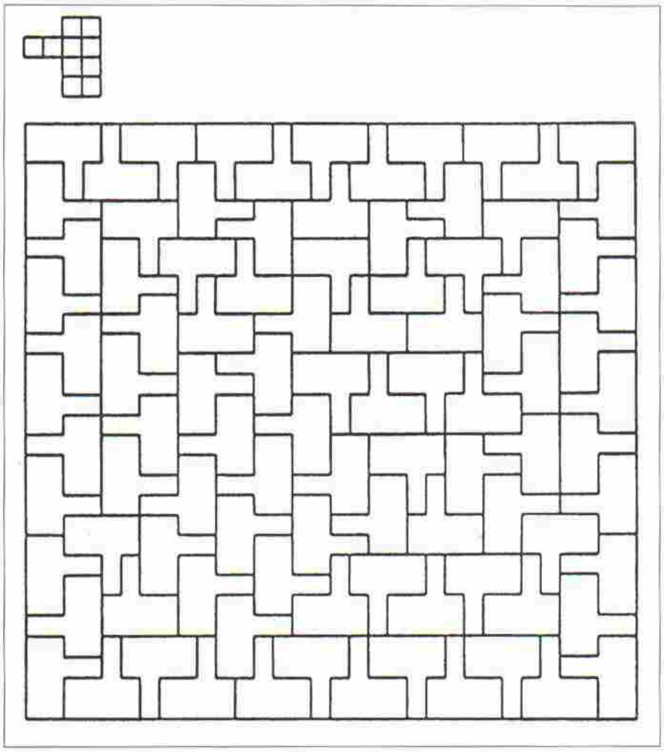

Figure 13. A dekomino of order 96.

The heptomino of order 76 in Figure 12 cannot tile its minimum rectangle with 180° rotational symmetry. This is also true of the dekomino in Figure 13 of order 96, whose minimum rectangle (the 30×32) was discovered by William Rex Marshall in 1991. In 1995, Marshall also found the minimum rectangles for the order 192 octomino (32×48) and the order 138 dekomino (30×46). These will be considered later.

Finally, Figure 14 shows the example of order 312 (Dahlke, 1988), although in this case it is not absolutely certain that no smaller number of copies of the octomino in question will form a rectangle.

No polyomino whose order is an odd number greater than 1 has ever been found, but the possibility that such polyominoes exist (with orders greater than 3) has not been ruled out.

The known *even* orders of polyominoes are all the multiples of 4, as well as the numbers 2, 10, 18, 50, and 138. Curiously, these even orders which are not multiples of 4 all exceed multiples of 8 by two. Whether there are other even orders, and what they might be, is still unknown. The smallest even order for which no example is known is order 6. Figure 15 shows one way in which six copies of a polyomino can be fitted together to form a rectangle, but the polyomino in question (as shown) actually has order 2. Michael Reid recently found a *heptabolo* (a figure made of seven congruent isosceles right triangles) of order 6, also shown in Figure 15.

The Golomb construction for polyominoes of order 4s gives its first new example, order 8, when s = 2. The underlying tiling concept of how to fit 8 congruent shapes together to form a rectangle is shown in Figure 16.

Although the shape used in Figure 16 is not a polyomino, the same concept can be realized using the 12-omino shown in Figure 17. (The shape used in Figure 16 is a *triabolo*!)

To show that there are infinitely many dissimilar polyominoes having order 8, we can form the family of polyominoes shown in Figure 18. For each integer $r \geq 1$, this construction produces a $(3r^2 + 6r + 3)$-omino of order 8, and clearly no two of these are similar.

It is also easy to show that none of these polyominoes can have order less than 8. The proof begins by ob-

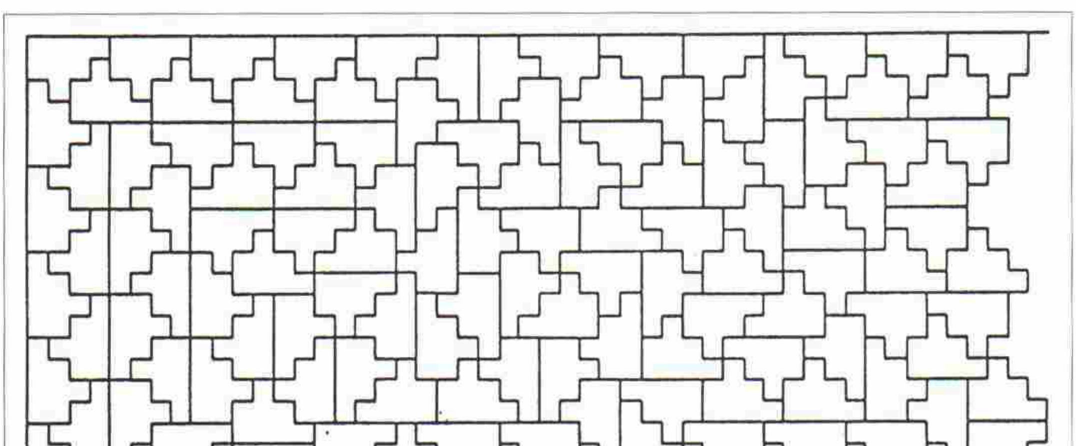

Figure 14. An example of order 312.

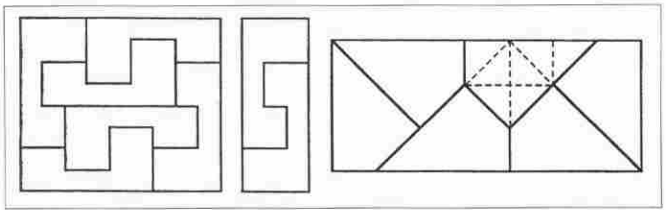

Figure 15. A 12-omino of order 2 which suggests an order-6 tiling, and Michael Reid's order-6 "heptabolo". (Is there any *polyomino* of order 6?)

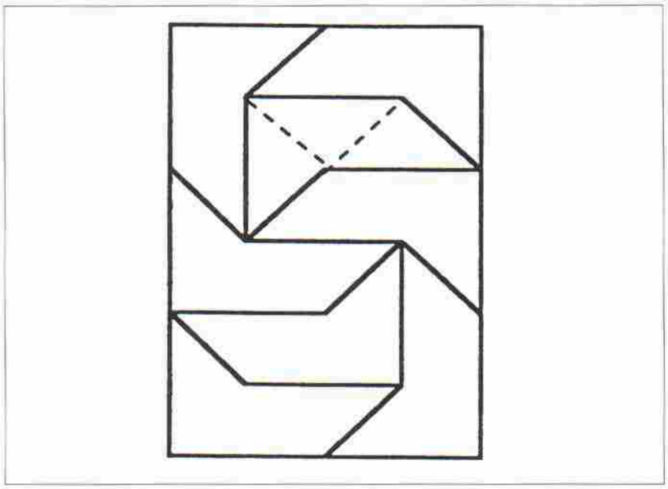

Figure 16. A rectangle formed from eight congruent pieces.

serving that only the "heel of the boot" can be in any *corner* of the rectangle to be tiled. Then the "toe" of the boot must be mated with the notch at the top-back of another boot. The quickest way to finish off the rectangle then requires eight copies of the polyomino.

In Figure 19, we see a construction for a polyomino of order $n = 4s$ for every $s = 1, 2, 3, 4, \cdots$. (Starting with a 2 by $4s - 2$ rectangle, we remove a single square from

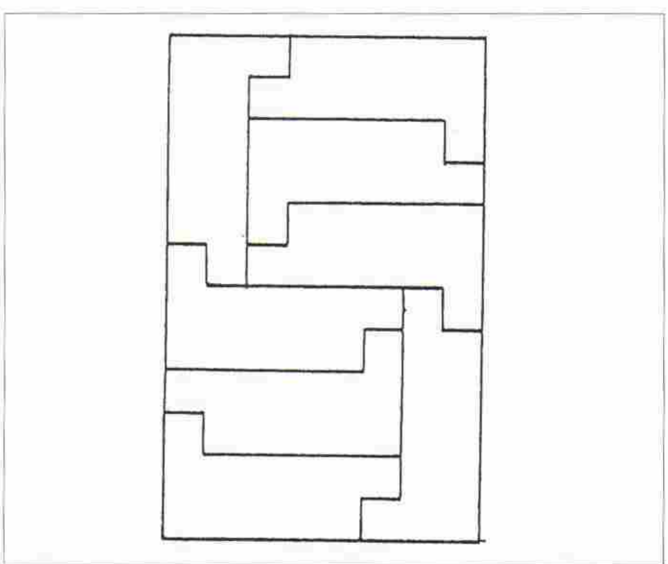

Figure 17. A polyomino of order 8.

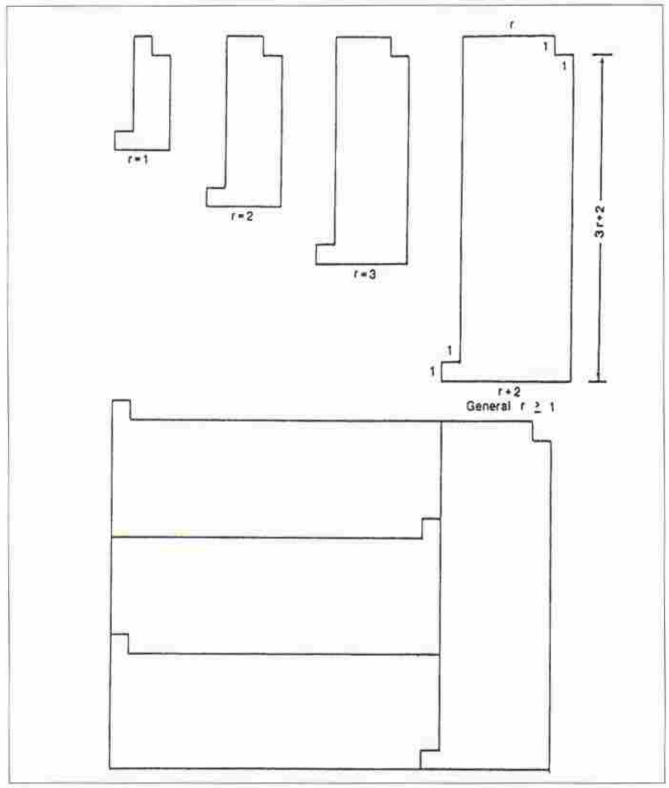

Figure 18. Dissimilar polyominoes of order 8, and how to stack them.

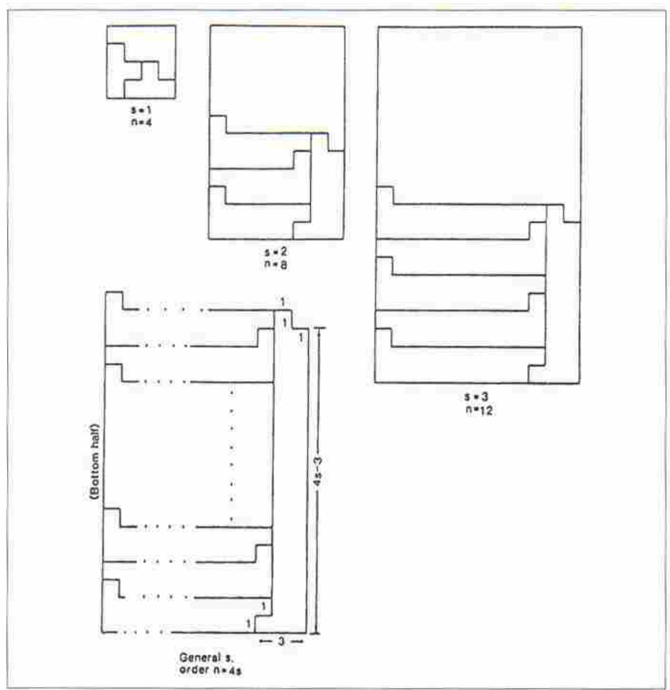

Figure 19. Polyominoes of order n=4s, for every positive integer s.

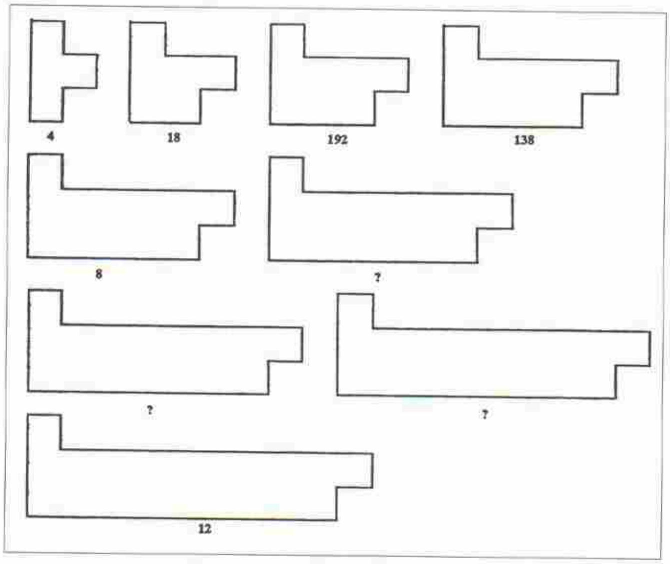

Figure 20. Infinite family of polyominoes. Does each one tile a rectangle? (The number below each figure is its order, if known.)

one corner and attach it as a "toe" at the opposite corner, to obtain the polyomino of order $4s$.)

The idea shown in Figure 18 can be applied not only to order $n = 8$, but to any order $n = 4s$, to obtain infinitely many dissimilar polyominoes of order $4s$. The general construction involving both r and s begins with a rectangle which is $(r + 1) \times (2s - 1)(r + 1)$, and moves a single 1×1 square from the top-back of the "boot" to become a "toe" at the opposite corner. (The proof that the resulting figure truly has order $n = 4s$ is analogous to the proof given for $n = 8$.)

As the reader is probably aware by now, the game polyominologists play is: given a polyomino, will it or won't it tile? In recent years, whenever I have publicized a specific polyomino whose ability to tile any rectangle was not yet decided, someone with a good computer program has come forth, usually within a year, with a rectangle-tiling solution. This time, I suggest an infinite family of polyominoes, the first several of which are known to tile rectangles, as is every fourth one throughout the family. Can you show that *any*, or *all*, of the others can tile rectangles? The family is illustrated in Figure 20. Two of the minimum rectangles, discovered in 1995 by William Rex Marshall, are shown in Figure 21.

We now return to the tiling of figures other than rectangles. If a polyomino has no order (i.e., if it cannot tile any rectangle), it may still be able to tile the entire plane, or various sub-regions of the plane, such as an infinite strip, or a bent strip. Such tilings are illustrated in Figure 22, using the X-pentomino, the F-pentomino, and the N-pentomino, respectively.

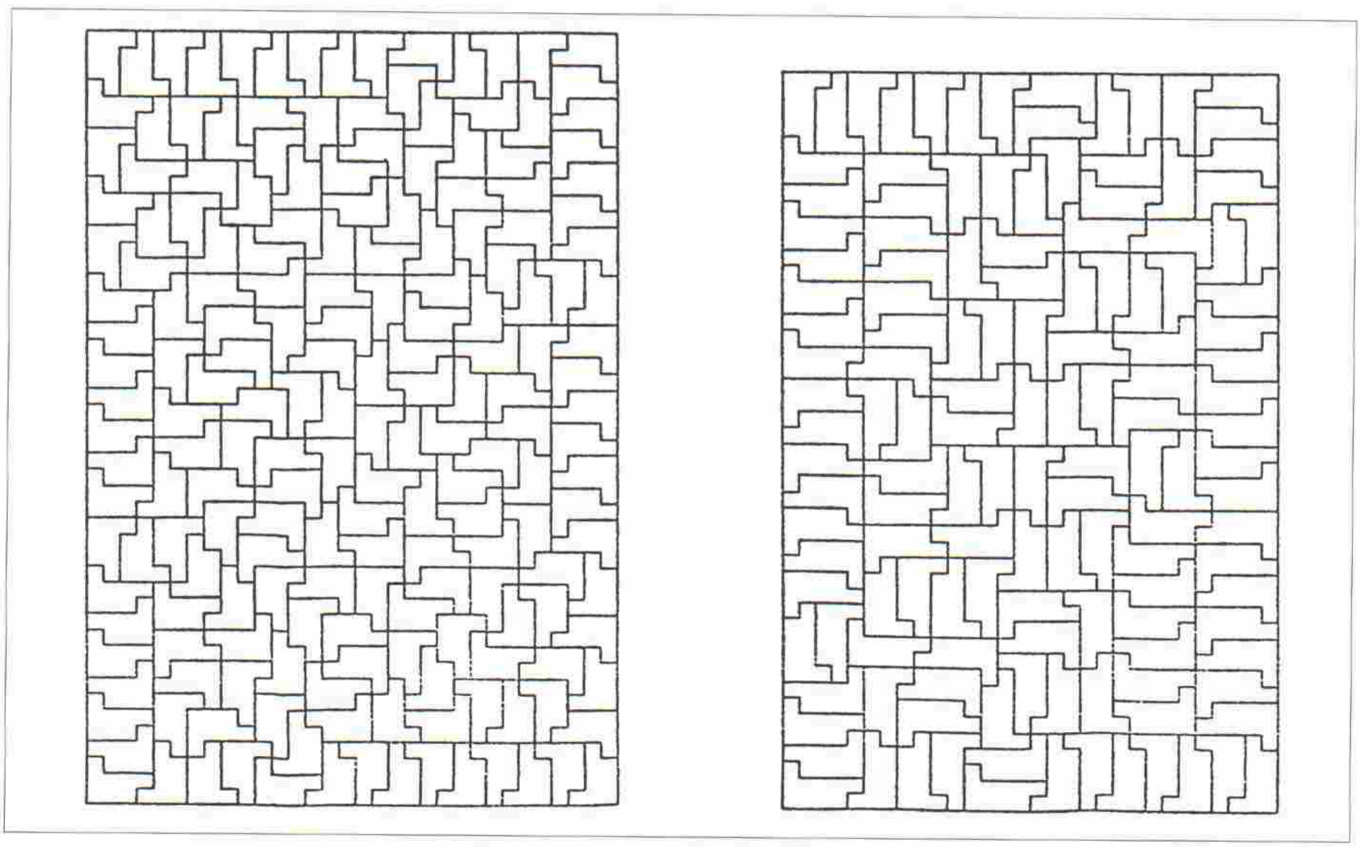

Figure 21. An octomino of order 192 and a dekomino of order 138.

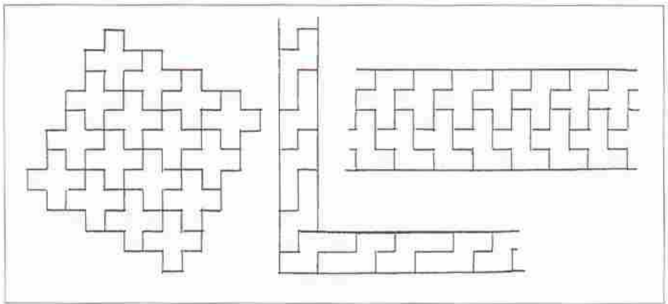

Figure 22. The X-pentomino tiles the plane; the F-pentomino tiles a strip; the N-pentomino tiles a bent strip.

In Figure 23, we see a tiling hierarchy for polyominoes (Golomb, 1966). A polyomino which can tile any of the regions specified in the hierarchy can also tile all the regions lower in the hierarchy. Thus, the "true category" of a polyomino is the *highest* box in the hierarchy which it can occupy. Most of this article has been concerned with polyominoes which occupy the highest box—i.e., they can tile rectangles. The "true categories" which are known to have members are: Rectangle, Bent Strip, Strip, Itself, Plane, and Nothing. Figure 24 shows an example of the category "Nothing". (The others have already been illustrated. The rep-tiles in Figure 4 illustrate the category "Itself.") For each of the other positions in the hierarchy, it is an open question whether any polyomino has that position as its "true category".

Most of the inclusion relations in Figure 23 are immediately obvious. To see that a polyomino which tiles a bent strip can tile both a quadrant and a straight strip, we first observe that bent strips (as in Figure 22) can be "nested" to cover a quadrant of the plane. We will show how to go from the bent strip to the straight strip after taking up one last important subject.

The most challenging unsolved problem about planar tiling involves shapes which can tile the infinite plane, but only non-periodically. R.M. Robinson was the first to find a finite (but very large) set of shapes such that unlimited copies of members of the set could be used to tile the infinite plane, but only non-periodically. He eventually reduced the size of the set to six. Independently, Roger Penrose found a set of 6 shapes with the nonperiodic-only tiling property, which he eventually reduced to a set of *two* shapes. I believe that a year-2000 version of Hilbert's list would include the question: Is there a *single* geometric shape S such that congruent copies of S can be used to tile the infinite plane, but only non-periodically?

To understand this question better, note that the answer is negative if instead of the plane we consider an infinite strip, even if we allow more than one tile shape to be used. To see this, consider a horizontal strip. Now choose any vertex on the upper edge, and trace a "jagged edge path" using the following rules: If there is a downward edge follow it. If not, move to the right. Continue to prefer "downward", and secondarily "rightward". Sometimes one may even be forced to go "upward", or "leftward", to trace around a protuberance on a tile, but clearly one will eventually arrive at the bottom edge of the strip, and one can easily obtain a uniform bound on the number of "moves" required in terms of the thickness of the strip, and the sizes and shapes of the tiles. This means that there are only a finite number of possible jagged paths, so there must be two different starting vertices giving the same jagged path. But then, if one takes copies of the region between these two identical paths and lays them end to end, one gets the desired periodic tiling of the strip. Further, we

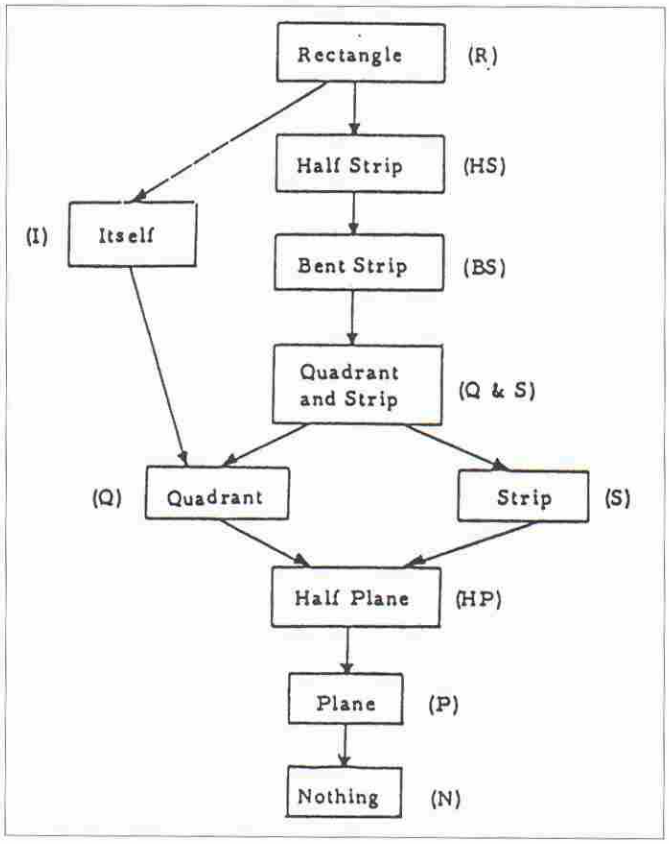

Figure 23. The hierarchy of tiling capabilities for polyominoes.

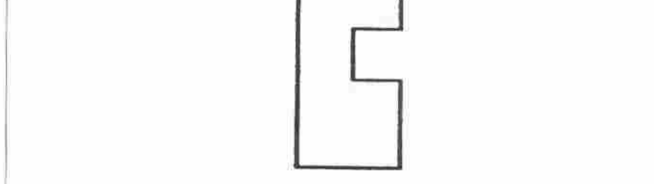

Figure 24. A 9-omino which cannot tile the plane.

see as promised, that polyominoes which can tile a bent strip can also tile a straight strip (applying the same argument to either one of the semi-infinite arms of the bent strip), and in fact the tiling can be made periodic.

In Figure 25, we see how the "P-pentomino" can be used to tile a quadrant non-periodically, by iterating the rep-tile subdivision of the P-pentomino into smaller copies of itself. We enlarge the picture, repeat the process, enlarge the picture, repeat the process, and "eventually" we have filled the first quadrant of the plane. Reflections in the coordinate axes can be used to cover the entire plane in a non-periodic manner. (This does not solve the previously mentioned problem, because the P-pentomino can also be used to tile the plane periodically.)

Not only the P-pentomino, but *every* polyomino rep-tile, can be shown to tile a quadrant. To show this, let J be any polyomino rep-tile, and let R be the smallest rectangle containing J with sides parallel to the grid-lines of J. We first prove the lemma that J must cover at least one of the four corners of its minimum rectangle R. Suppose not. In Figure 26, we see a polyomino K which occupies none of the corners of its minimum rectangle. Consider the left-most square of K along the bottom of its minimum rectangle, indicated with an asterisk. If K were a rep-tile, we could divide it into congruent replicas, and we could iterate this process until each replica was so small that it would fit within a single square of the original grid. At this stage, consider the replica k of K covering the lower left corner of the square "*". It fits entirely within that square, and fills a corner of it, so

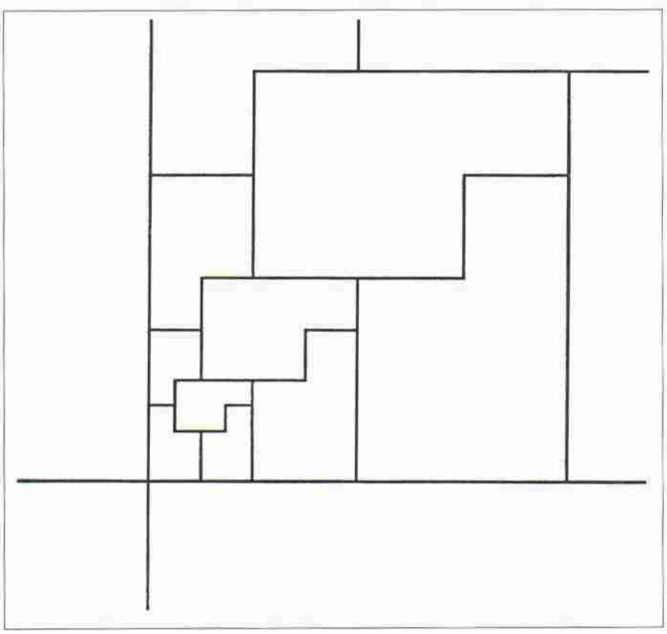

Figure 25. Rep-tilic nonperiodic tiling of a quadrant with congruent copies of the P-pentomino.

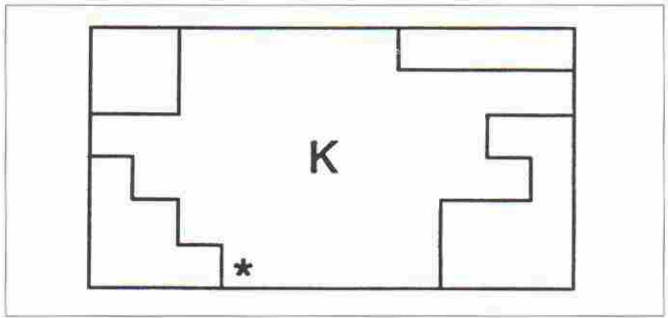

Figure 26. A polyomino inscribed in its minimum rectangle.

clearly k occupies a corner of its minimum rectangle. But the same would be true of K.

Now, knowing that J must occupy at least one corner of its minimum rectangle, put such a corner square of J into the lower left corner of the first quadrant of the plane. Now, perform the rep-tilic subdivision of J, enlarge back to the original scale, and continue to subdivide and enlarge (as we did with the P-pentomino in Figure 25) until we have filled up the entire first quadrant. (We do not need to invoke König's lemma to assert that a limiting tiling exists. The method is constructive. Given any radius R, however large, we can specify the tiling of the first quadrant out to distance R from the origin, in such a way that this tiling will not change as R is increased. This may involve looking only at every m^{th} iteration of the rep-tilic subdivision, if the J nearest the origin is moved by a single iteration.)

The proofs that a polyomino which tiles a bent strip will tile a straight strip, and that a polyomino which tiles itself (i.e., a rep-tile) will tile a quadrant, were given in Golomb (1966).

Finally, returning to the subject of non-periodic tilings of the plane, the natural question to ask in our present context is whether there exist "non-periodizable" tilings using polyominoes. The best current result is the set of *three* shapes, shown in Figure 27, which can be used to tile the plane, but not periodically. (This set was recently discovered by Penrose, who writes that he derived these from a tiling set by Robert Ammann.) Penrose's aperiodic tilings also have connections to "quasicrystals", which have been of major interest to chemists in recent years.

Arguments which show that certain sets of "pieces" can tile the plane, but not periodically, are both clever and subtle (see Grünbaum and Shephard, 1987, Chapter 10), and undoubtedly lie outside the range of anything Hilbert ever contemplated. But Hilbert's insight which led him to include a problem about tilings and packings in his famous List was profound. This is a subject which is accessible to amateurs but lies close to the very heart of mathematics, and continues to provide a seemingly

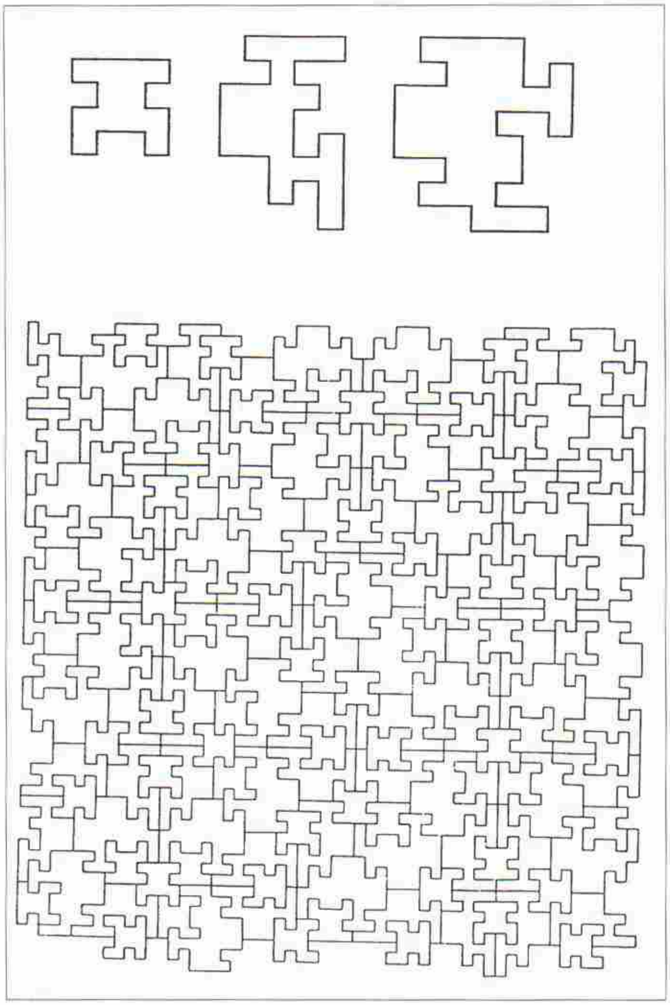

Figure 27. Roger Penrose's set of three polyominoes which can be used to tile the plane, but not periodically.

inexhaustible supply of intriguing and provocative questions.

NOTE: This material is based on Chapter 8 of *Polyominoes—Revised Edition*, by Solomon W. Golomb, Princeton University Press, 1994. It is one of the new chapters added to the text of the original edition of *Polyominoes*, which was published in 1965 by Charles Scribner's Sons.

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To delete, simultaneously press
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F1 Delete letter
F2 Delete word
F3 Delete line
F4 Delete page
F5 Delete file
F6 Delete subdirectory
F7 Reformat hard drive
F8 Electrocute user
F9 Detonate building
F10 Advanced options on next screen

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